

## Review of Linear Expressions and Equations

Skills involving linear equations can be divided into the following groups:

- Simplifying algebraic expressions.
- Linear expressions.
- Solving linear equations.
- Graphs of linear functions.

### Simplifying Algebraic Expressions

Algebraic expressions are a combination of numbers and variables. Here are examples of some basic algebraic expressions.

$$3x + 4$$

$$5x - 2y + 6$$

$$x^2 + 5x - 4$$

$$x^4 - 16$$

$$x^2 + 5xy + 6y^2$$

The **terms** of an algebraic expression are those quantities that are separated by addition.

*Example:* List the terms of the algebraic expression  $4x^2 - 3x + 6$

Identification/Analysis	Terms are identified by the addition operation. Write the expression using addition..	
Solution	$4x^2 - 3x + 6$ $4x^2 + (-3x) + 6$ $4x^2, -3x \text{ and } 6$	Write the expression using addition. Subtraction can be written in terms of addition: $a - b = a + (-b)$ Identify the terms in the expression.

The **coefficient** of a term is its numerical factor.

*Example:* Determine the coefficients of the following terms.  $5x^2$ ,  $\sqrt{2}xy^3$ ,  $\frac{x^2y^3}{4}$ ,  $a^2b$ .

Identification/Analysis	Factors are quantities that are multiplied together. Think of each term as a product of a number (the coefficient) and variables.	
Solution	$5x^2$ $\sqrt{2}xy^2$ $\frac{x^2y^3}{4} = \frac{1}{4}x^2y^3$ $a^2b = 1 \cdot a^2b$	The coefficient is 5. The coefficient is $\sqrt{2}$ . The coefficient is $\frac{1}{4}$ . The coefficient is 1.

**Like terms** (also known as **common terms** or **similar terms**) are terms that have the same variable factors.

*Example:* Determine the like terms in the following list.  $5x^2$ ,  $-3y^2$ ,  $8xy$ ,  $2x^2y$ ,  $2xy$ ,  $7x^2y$ ,  $10x$ , and  $9y^2$

Identification/Analysis	Compare the variable factors in two terms. For example, when considering the term $7x^2y$ , the variable factors are $x^2y$ . (Note that 7 is the coefficient.)	
Solution	$-3y^2$ and $9y^2$ are like terms. $8xy$ and $2xy$ are like terms. $2x^2y$ and $7x^2y$ are like terms.	The variable factors are $y^2$ in both terms. The variable factors are $xy$ in both terms. The variable factors are $x^2y$ in both terms.

Note that there are two terms left over:  $5x^2$  and  $10x$ . These two terms are not like terms because their variable factors are not the same ( $x^2$  and  $x$  respectively).

The properties of the real numbers are used when working with algebraic expression. Consider the following examples.

Properties of Real Numbers	
Property	Example
Commutative Property of Addition $a + b = b + a$	$14x + 5x^2 = 5x^2 + 14x$
Commutative Property of Multiplication $a \cdot b = b \cdot a$	$x^3 \cdot 7 = 7x^3$
Associative Property of Addition $(a + b) + c = a + (b + c)$	$(3x^2 + 4x) + 7x = 3x^2 + (4x + 7x)$
Associative Property of Multiplication $(a \cdot b) \cdot c = a \cdot (b \cdot c)$	$5(4x) = (5 \cdot 4)x$
Distributive Property $a(b + c) = a \cdot b + a \cdot c$ Can also be written as: $(b + c)a = b \cdot a + c \cdot a$	$5(x + 2) = 5x + 5 \cdot 2$  $(4 + y)x = 4 \cdot x + y \cdot x$

The distributive property allows us to add or subtract like terms. The adding (or subtracting) of s like terms is known as **combining like terms**.

*Example:* Combine the like terms  $3x^2 + 5x^2$

Identification/Analysis	The two terms being added are like terms. Their variable factors are both $x^2$ . Therefore it is possible to combine them using the distributive property.	
Solution	$3x^2 + 5x^2$ $(3+5)x^2$ $8x^2$	Use the distributive property.

*Example:* Combine the like terms:  $3x^2 + 4x - 2x^2 + 3x - 4$

Identification/Analysis	There are five terms in this expression. Identify the like terms.	
Solution	$3x^2 + 4x - 2x^2 + 3x - 4$ $3x^2 - 2x^2 + 4x + 3x - 4$ $(3 - 2)x^2 + (4 + 3)x - 4$ $x^2 + 7x - 4$	$3x^2$ and $-2x^2$ are like terms as are $4x$ and $3x$ . Use the commutative property of addition to bring like terms together. Use of the distributive property to combine like terms.

To **simplify** an algebraic expression with parenthesis use the distributive property, then combine like terms.

*Example:* Simplify  $2(5x - 1) + 14x$

Identification/Analysis	Before any terms can be combined in this expression the distributive property must be used in order to remove the parenthesis.	
Solution	$2(5x - 1) + 14x$ $10x - 2 + 14x$ $24x - 2$	Use of the distributive property. Combine like terms.

*Example:* Simplify  $5(x^2 + 2x) - x(2x - 4)$

Identification/Analysis	There are two sets of parenthesis so the distributive property will be used twice.	
Solution	$5(x^2 + 2x) - x(2x - 4)$ $5x^2 + 10x - 2x^2 + 4x$ $5x^2 - 2x^2 + 10x + 4x$ $3x^2 + 14x$	Use of the distributive property. Use of the commutative property of addition to bring like terms together. Combine like terms.

## Linear Expressions

The **degree** of a term is the sum of the exponents on the variable factors.

*Example:* Determine the degree of  $5x^2y$ .

Identification/Analysis	Identify the exponents on the variable factor, to determine the degree of the terms.	
Solution	$5x^2y$ $x^2$ and $y$ 2 and 1 $2 + 1$ or 3	Identify the variable factors. Identify the exponents on the variable factors. Add the exponents to determine the degree of the term.

Note: The degree of a constant (e.g. 6) is 0, since  $6 = 6x^0$ .

*Example:* Determine the degree of the constant 8.

Identification/Analysis	Since the degree of a term is based on the variable factor, rewrite the term using a variable (recall that $x^0 = 1$ ).	
Solution	8 $8x^0$ 0	Rewrite 8 with a variable factor. The degree of the term.

A **linear expression** is a sum (or difference) of terms in which the highest degree of any term is 1.

*Example:* Determine if the expression  $3x - 4$  is linear.

Identification/Analysis	To determine whether or not the expression is linear we must determine the degree of each term.	
Solution	$3x$ and $-4$ 1 and 0 The expression is linear.	List the two terms of the expression. Identify the degree of $3x$ and $-4$ respectively. Since the highest degree is 1, the expression is linear.

*Example:* Determine whether the expression  $3(x - 4) - 2(x - 6)$  is linear.

Identification/Analysis	Since there are parenthesis, before determining the degree of each term, simplify the expression.	
Solution	$3(x - 5) - 2(x - 6)$ $3x - 15 - 2x + 12$ $x - 3$ $x$ and 3 1 and 0 The expression is linear.	Use of distributive property. Combine like terms. List the terms. Identify the degree of $x$ and 3 respectively. Since the highest degree is 1, the expression is linear.

*Example:* Determine if the expression  $3x^2 - 4x + 7$  is linear.

Identification/Analysis	To determine whether or not the expression is linear, determine the degree of each term.	
Solution	$3x^2$ , $-4x$ , and, 7 2, 1 and 0 The expression is <b>not</b> linear.	List the three terms of the expression. Identify the degree of $3x^2$ , $-4x$ , and, 7 respectively. The highest degree is greater than 1, so the expression is <b>not</b> linear.

## Solving Linear Equations

A linear equation (in one variable  $x$ ) is an equation that can be written in the form  $ax + b = 0$ .

Two properties are used to solve linear equations.

**Addition Property of Equality:** The same number (or algebraic expression) can be added to both sides of an equation without changing the equation's solution set.

**Multiplication Property of Equality:** The same nonzero number may multiply both sides of an equation without changing the equation's solution set.

*Example:* Solve the equation.  $x - 4 = 7$

Identification/Analysis	This is a linear equation since the variable, $x$ , is raised to the first power. Solve for $x$ .	
Solution	$x - 4 = 7$ $x - 4 + 4 = 7 + 4$ $x = 11$	Add 4 to both sides. Simplify. There is one solution.
Check	$(11) - 4 = 7$ $7 = 7$	Substitute $x = 11$ into the original equation and simplify the results. The statement is true so $x = 11$ is the solution.

*Example:* Solve the equation.  $5y = 40$

Identification/Analysis	This is a linear equation since the variable, $y$ , is raised to the first power. Solve for $y$ .	
Solution	$5y = 40$ $\frac{5y}{5} = \frac{40}{5}$ $y = 8$	Divide by 5. Note: Dividing by 5 is equivalent to multiplying by $1/5$ . There is one solution.
Check	$5(8) = 40$	Substitute $x = 11$ into the original equation and simplify the results. The statement is true so $x = 8$ is the solution

*Example:* Solve the equation.  $3x + 2 = 17$

Identification/Analysis	This is a linear equation since the variable, $x$ , is raised to the first power. Solve for $x$ .	
Solution	$3x + 2 = 17$ $3x + 2 - 2 = 17 - 2$ $3x = 15$ $\frac{3x}{3} = \frac{15}{3}$ $x = 5$	<p>Subtract 2 from both sides.</p> <p>Divide both sides by 3.</p> <p>There is one solution.</p>
Check	$3(5) + 2 = 17$ $15 + 2 = 17$ $17 = 17$	<p>Substitute <math>x = 5</math> into the original equation and simplify the results.</p> <p>The statement is true so <math>x = 5</math> is the solution.</p>

The previous examples have all been in the most basic form. As the problems become more involved you may want to follow the procedure outlined below.

### Steps used to Solve a Linear Equation

1. Simplify the algebraic expressions on each side of the equation (remove grouping symbols and combine like terms).
2. Collect all the terms with the variable on one side of the equation and all the constants (numerical terms) on the other side. This isolates the variable.
3. Divide both sides by the coefficient of the variable. (Multiplying by the reciprocal of the coefficient of the variable gives the same result.)
4. Check or verify the solution by substituting the value into the original equation.

*Example:* Solve the equation.  $2(x-4)-5x+2=-3$

Identification/Analysis	This is a linear equation since the variable, $x$ , is raised to the first power. Solve for $x$	
Solution	$2(x-4)-5x+2=-3$ $2x-8-5x+2=-3$ $-3x-6=-3$ $-3x-6+6=-3+6$ $-3x=3$ $\frac{-3x}{-3}=\frac{3}{-3}$ $x=-1$	<p>Use the distributive property.</p> <p>Combine like terms.</p> <p>Collect the constants on one side, so add 6 to both sides.</p> <p>Simplify.</p> <p>Divide both sides by <math>-3</math>, the coefficient of <math>x</math>.</p> <p>There is one possible solution.</p>
Verification	$2(x-4)-5x+2=-3$ $2(-1-4)-5(-1)+2=-3$ $2(-5)-5(-1)+2=-3$ $-10+5+2=-3$ $-3=-3$	<p>Original Equation</p> <p>Substitute <math>-1</math> for <math>x</math> in the original equation.</p> <p>Simplify inside the parentheses.</p> <p>Perform multiplication operations.</p> <p>The statement is true so <math>x=-1</math> is a solution.</p>

*Example:* Solve the equation.  $5(2z-8)-2=5(z-3)+3$  (Verification will be omitted for this problem.)

Identification/Analysis	This is a linear equation since the variable, $z$ , is raised to the first power. Solve for $z$ .	
Solution	$5(2z-8)-2=5(z-3)+3$ $10z-40-2=5z-15+3$ $10z-42=5z-12$ $10z-42-5z=5z-12-5z$ $5z-42=-12$ $5z-42+42=-12+42$ $5z=30$ $z=6$	<p>Use the distributive property.</p> <p>Combine like terms.</p> <p>Subtract <math>5z</math> from both sides to collect the variables on one side.</p> <p>Combine like terms.</p> <p>Add 42 to both sides to collect the constants on the opposite side of the variables.</p> <p>Divide both sides by 5.</p> <p>Check the solution on your own.</p>

*Example:* Solve the equation.  $\frac{3x}{4} - 3 = \frac{x}{2} + 2$  (Verification will be omitted for this problem.)

Identification/Analysis	This is a linear equation involving fractions. Begin by multiplying both sides of the equation by the least common denominator.	
Solution	$\frac{3x}{4} - 3 = \frac{x}{2} + 2$ $4\left(\frac{3x}{4} - 3\right) = \left(\frac{x}{2} + 2\right) 4$ $4 \cdot \frac{3x}{4} - 4 \cdot 3 = 4 \cdot \frac{x}{2} + 4 \cdot 2$ $3x - 12 = 2x + 8$ $3x - 12 - 2x = 2x + 8 - 2x$ $x - 12 = 8$ $x - 12 + 12 = 8 + 12$ $x = 20$	<p>Multiply both sides by 4</p> <p>Use the distributive property.</p> <p>Simplify the fractions.</p> <p>Subtract <math>2x</math> from both sides to collect the variables on one side.</p> <p>Combine like terms.</p> <p>Add 12 to both sides to collect the constants on the opposite side of the variables.</p> <p>Combine like terms.</p>

### Linear Equations can have one, none, or many solutions.

Thus far all of our examples have had a single solution. It is also possible that a linear equation will have no solution or all real numbers as the solution.

When solving an equation with no solution, the variable will be eliminated as you attempt to collect the variables on one side and the constants on the other side. This will result in a false statement, such as  $5 = 8$  or  $3 = -1$ . Thus, when you obtain a false statement there are no solutions to the linear equation.

When solving an equation with all real numbers as the solution, you will eliminate the variable as you collect the variables on one side and the constants on the other side. Instead of a false statement you will obtain a true statement, such as  $7 = 7$ . Thus, when you obtain a true statement, an identity, any real number will satisfy the linear equation.

*Example:* Solve the equation.  $2(2x - 1) - 3 = 4(x + 2)$

Identification/Analysis	This is a linear equation in the variable $x$ . Solve for $x$ .	
Solution	$2(2x - 1) - 3 = 4(x + 2)$ $4x - 2 - 3 = 4x + 8$ $4x - 5 = 4x + 8$ $4x - 5 - 4x = 4x + 8 - 4x$ $-5 = 8$ <p>No Solution</p>	<p>Use the distributive property. Combine like terms. Subtract <math>4x</math> from both sides. The variable is eliminated. The statement is false.</p>

To verify there is no solution pick any real number for  $x$ . Substitute the value into the original equation.

Verification	$2(2(5) - 1) - 3 = 4(5 + 2)$ $2(10 - 1) - 3 = 4(7)$ $2(9) \neq 28$	<p>Substitute 5 for <math>x</math> in the original equation and simplify the results. Choose a second <math>x</math> value like <math>x = 0</math> and check again.</p> <p>The statement is false which verifies <math>x = 5</math> is not a solution. Any value you choose would result in a false statement so there are no solutions.</p>
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*Example:* Solve the equation.  $5y - 3(y - 1) = 2(y + 3) - 3$

Identification/Analysis	This is a linear equation in the variable $y$ . Solve for $y$ .	
Solution	$5y - 3(y - 1) = 2(y + 3) - 3$ $5y - 3y + 3 = 2y + 6 - 3$ $2y + 3 = 2y + 3$ $2y + 3 - 2y = 2y + 3 - 2y$ $3 = 3$ $\{y \mid y \text{ is a real number}\}$	<p>Use the distributive property. Combine like terms Subtract <math>2y</math> from each side. The variable <math>y</math> is eliminated. The statement is true. Any real number will satisfy the equation. The solution is the set of real numbers. (<math>y \in \mathbb{R}</math>)</p>

To verify any real number is a solution pick any real number for  $x$ . Substitute the value into the original equation. This is shown for  $y = 4$  on the next page.

Verification	$5(4) - 3(4-1) = 2(4+3) - 3$ $20 - 9 = 2(7) - 3$ $11 = 11$	<p>Substitute 4 for <math>y</math> in the original equation and simplify the results. Choose a second <math>y</math> like <math>y = 0</math> or <math>y = 1</math> and check again.</p> <p>The statement is true which verifies <math>y = 4</math> is a solution.</p> <p>Any value you choose would result in a true statement so <math>y</math> is any real number.</p>
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## Graphs of Linear Functions

The graph of a linear function  $f(x) = mx + b$  is a line with slope  $m$  and  $y$ -intercept  $b$ . This format is called **slope-intercept** form. It is also written  $y = mx + b$ .

The slope of a line through the points  $(x_1, y_1)$  and  $(x_2, y_2)$  is given by the formula  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

*Example:* Determine the slope of the line through the points  $(5, 8)$  and  $(13, -4)$

Identification/Analysis	When substituting into the slope formula it does not matter which point is chosen to be $(x_1, y_1)$ . In this example we will let $(x_1, y_1)$ be $(13, -4)$ and we will let $(x_2, y_2)$ be $(5, 8)$ .	
Solution	$m = \frac{8 - (-4)}{5 - 13}$ $m = \frac{12}{-8}$ $m = -\frac{3}{2}$	<p>Substitute into the slope formula.</p> <p>Perform the subtraction operation in the numerator and denominator.</p> <p>Reduce the fraction.</p>

*Example:* Write the linear equation  $3x + 4y = 6$  in slope-intercept form.  $y = mx + b$  or  $f(x) = mx + b$

Identification/Analysis	To write the equation in slope-intercept form solve the equation for $y$ .	
Solution	$3x + 4y = 6$ $4y = -3x + 6$ $y = \frac{-3x + 6}{4}$ $y = -\frac{3}{4}x + \frac{3}{2} \quad \text{or}$ $f(x) = -\frac{3}{4}x + \frac{3}{2}$	<p>Subtract <math>3x</math> from both sides. Divide both sides by 4. To obtain slope intercept form, separate the right side into two terms and simplify fractions if possible. Slope intercept form using <math>y = mx + b</math></p> <p>Slope intercept form using function notation</p>

*Example:* Determine the slope intercept form for the equation of the line through the points  $(-12, -6)$  and  $(9, 8)$ .

Identification/Analysis	To find the equation of the line determine the slope, $m$ , and the $y$ -intercept, $b$ .	
Solution	$\frac{-6 - 8}{-12 - 9} = \frac{-14}{-21} = \frac{2}{3}$ $8 = \frac{2}{3}(9) + b$ $8 = 6 + b$ $2 = b$ $f(x) = \frac{2}{3}x + 2$	<p>Determine the slope. Let <math>(x_2, y_2)</math> be <math>(-12, -6)</math> and <math>(x_1, y_1)</math> be <math>(9, 8)</math>. Remember to reduce the fraction.</p> <p>Determine the <math>y</math>-intercept. Substitute the slope of <math>\frac{2}{3}</math> and the point <math>(9, 8)</math> into the linear function formula. Solve for <math>b</math>. Note: Either point can be used to find <math>b</math>.</p> <p>State the equation using function notation.</p>